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Improved Ratio Type Generalized Class of Estimators in Two Phase Adaptive Cluster Sampling

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Abstract


In this paper, an improved ratio type class of estimators in the two phase Adaptive Cluster Sampling (ACS) design under the transformed population approach has been proposed. The generalized expressions of Bias and Mean Squared Error (MSE) have been obtained up to the first order of approximation. New member estimators are developed from the proposed class and their performance against competing existing estimators are evaluated using various empirical studies. The novelty of this design and the new estimators developed therein are further demonstrated using a real data study where the new developed estimators are used to estimate the average number of thorny plants in plateaus of Western Ghats of Sahyadri from Goa to Varandha Ghat (Bhor, Maharashtra, India).


Keywords: Adaptive cluster double sampling, Two phase adaptive cluster sampling, Transformed population approach, Regression type estimators, Ratio estimators.


1 | Introduction

Thompson [1] introduced the Adaptive Cluster Sampling (ACS) design to deal with populations that were hidden clustered type. For instance, the cases of COVID-19, were a perfect example of the hidden clustered type population during the initial days. As a result, some researchers used the ACS design on COVID-19 [2], [3]. Apart from these contemporary applications, the ACS design was used by various researchers soon after it was proposed by Thompson [1], [4]–[7].

The work on estimators development in the ACS design comprehensively started after the proposal of transformed population approach by Dryver and Chao [8] where they proposed that ACS can be considered as Simple Random Sampling Without Replacement (SRSWOR) if we consider the network means for

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estimation (for details see [8]). Since then, many ratio type and product type estimators have been proposed in the ACS design. Dryver and Chao [8] proposed the ratio type estimator using single auxiliary variable motivated by Cochran [9]. Chutiman [10] proposed some transformed ratio type estimators using some known parameters of the auxiliary variable. Yadav et al. [11] proposed some improved ratio type estimators in the ACS design. Qureshi et al. [12] proposed a generalized ratio type estimators using some robust measures.

In practice, there are many situations where the mean of auxiliary variable μ_x is not known and in this case the ratio and product type estimators cannot be used. To resolve this problem, the two phase sampling design was proposed where the researchers first draw an initial sample from the population also called as the first phase sample and estimate μ_x and then the second phase sample is drawn and observations on both x and y are observed and estimates are calculated. Such a situation may also arise in practice in the case of rare type populations and to solve this problem Félix-Medina and Thompson [13] proposed the adaptive cluster double sampling. Younis and Shabbir [14] used the Adaptive cluster double sampling or the two phase ACS design under the transformed population approach with multi-auxiliary variables.

Raghav et al. [15] proposed the ratio type, product type and a generalized robust ratio type estimator using single auxiliary variable in the two phase ACS design under the transformed population approach in both strategies. It is worth noting that the minimum Mean Squared Error (MSE) of the generalized robust ratio type estimator proposed by Raghav et al. [15] is equal to the minimum MSE of the regression estimator. A comprehensive literature review of the Two phase ACS design under the transformed population approach reveals that a more comprehensive study of estimators using single auxiliary variable which may provide a lower MSE than the regression type estimator is needed. Therefore, in this paper we propose an improved ratio type class of estimators using single auxiliary variable better than the regression type estimator motivated from Gupta and Shabbir [16]. Further, we explore the inclusion of one more known parameter of auxiliary variable motivated from Singh et al. [17]. The presentation sequence of the paper is presented below.

For new readers we recommend Thompson [1], Dryver and Chao [8], Félix-Medina and Thompson [13] and Raghav et al. [15] for getting acquainted with the methodology and terminologies of the ACS and the two phase ACS design under transformed population approach. The proposed improved ratio type generalized class is presented in Section 2 along with some new developed member estimators and the derivation of Bias and MSE up to first order of approximation. In Section 3, using various empirical studies we compare the performance of the new developed member estimators of the proposed class with related existing competing estimators. In Section 4, the two phase ACS design along with the new developed estimators are applied on a real data to estimate the average number of thorny plants in plateaus of Western Ghats of Sahyadri from Goa to Varandha Ghat (Bhor, Maharashtra, India) Latpate and Krishnasagar [18]. In Section 5, the results and concluding remarks have been presented. The related existing competing estimators have been provided in the Appendix.

2| Proposed Improved Ratio Type Generalized Class in Two Phase ACS Design

The aim of this research paper is to develop an improved ratio type class of estimators which can provides lower MSE than the regression type estimators. Therefore we propose an improved ratio type class of estimator by making Gupta and Shabbir [16] class more generalized and including one more parameter of auxiliary variable taking motivations from Olkin [19], Qureshi and Hanif [20], Singh et al. [17], Raghav et al. [15] and Singh et al. [21] as follows:

$$t_{IR} = \left(k_1 \bar{w}_y + k_2 (\mu_x^* - \bar{w}_x) \right) \left(\frac{\alpha \mu_x^* + C\beta}{\theta(\alpha \bar{w}_x + C\beta) + (1 - \theta)(\alpha \mu_x^* + C\beta)} \right)^g. \quad (1)$$

where k_1 and k_2 are optimised such that the MSE of the proposed class t_{IR} is minimum. α , C and β may assume different known values of parameters of the auxiliary variable namely coefficient of kurtosis, skewness,

Hodges-Lehmann, coefficient of variation among others and θ and g takes different constant values to yield different classes. μ^* is the estimated mean of auxiliary variable X obtained from first phase sample. From this proposed class, we develop some new estimators as follows:

$$\begin{aligned} \text{I. } t_{IR_1} &= \left(k_{1\text{opt}} \bar{w}_y + k_{2\text{opt}} (\mu_x^* - \bar{w}_x) \right) \left(\frac{\mu_x^* + C_{w_x}^2}{\bar{w}_x + C_{w_x}^2} \right). \\ \text{II. } t_{IR_2} &= \left(k_{1\text{opt}} \bar{w}_y + k_{2\text{opt}} (\mu_x^* - \bar{w}_x) \right) \left(\frac{\mu_x^* + C_{w_x}^2 \beta_2(w_x)}{\bar{w}_x + C_{w_x}^2 \beta_2(w_x)} \right). \\ \text{III. } t_{IR_3} &= \left(k_{1\text{opt}} \bar{w}_y + k_{2\text{opt}} (\mu_x^* - \bar{w}_x) \right) \left(\frac{\mu_x^* + \beta_1(w_x) \beta_2(w_x)}{\bar{w}_x + \beta_1(w_x) \beta_2(w_x)} \right). \end{aligned}$$

Using the following error terms, we derive the expressions of Bias and MSE of the proposed class t_{IR} .

$$e_{w_y} = \frac{\bar{w}_y}{\mu_y} - 1, e_{w_x} = \frac{\bar{w}_x}{\mu_x} - 1, e'_{w_x} = \frac{\mu_x^*}{\mu_x} - 1,$$

$$E(e_{w_y}^2) = f C_{w_y}^2, E(e_{w_x}^2) = f C_{w_x}^2, E(e'_{w_x}{}^2) = f' C_{w_x}^2,$$

and

$$\begin{aligned} E(e_{w_y} e_{w_x}) &= f \rho_{w_x w_y} C_{w_x} C_{w_y}, E(e_{w_y} e'_{w_x}) = f' \rho_{w_x w_y} C_{w_x} C_{w_y}, E(e_{w_x} e'_{w_x}) = \\ f' C_{w_x}^2 \text{ where } C_{w_x}^2 &= \frac{s_{w_x}^2}{\mu_x^2}, C_{w_y}^2 = \frac{s_{w_y}^2}{\mu_y^2}, S_{w_x}^2 = \frac{1}{N-1} \sum_{i=1}^N (w_{x_i} - \mu_x)^2, S_{w_y}^2 = \\ \frac{1}{N-1} \sum_{i=1}^N (w_{y_i} - \mu_y)^2, \rho &= \frac{s_{w_x w_y}}{s_{w_x} s_{w_y}}, S_{w_x w_y} = \frac{1}{N-1} \sum_{i=1}^N (w_{x_i} - \mu_x)(w_{y_i} - \mu_y), f = \frac{1}{n} - \\ \frac{1}{N} \text{ and } f' &= \frac{1}{n'} - \frac{1}{N}. \end{aligned}$$

We write Eq. (1) in error terms described above as

$$t_{IR} = (k_1 \mu_y + k_1 \mu_y e_{w_y} + k_2 \mu_x e'_{w_x} - k_2 \mu_x e_{w_x}) \left((1 + \Psi' e'_{w_x})^g (1 + \Psi' (e'_{w_x} + \theta(e_{w_x} - e'_{w_x})))^{-g} \right), \quad (2)$$

$$\text{where } \Psi' = \frac{\alpha \mu_x}{\alpha \mu_x + C \beta}.$$

Simplifying Eq. (2) and subtracting μ_y from both sides, we get

$$\begin{aligned} t_{IR} - \mu_y &= -\mu_y + k_1 \mu_y \left\{ 1 + e_{w_y} - g \Psi' \theta e_{w_x} + g \Psi' \theta e'_{w_x} + \frac{g(g+1)}{2} \Psi^2 \theta^2 e_{w_x}^2 + \right. \\ &(g(g+1) \Psi^2 + g(g+1) \Psi^2 \theta^2 - g(g+1) \Psi^2 \theta - g^2 \Psi^2 + g^2 \Psi^2 \theta + g(g- \\ &1) \Psi^2) e'_{w_x}{}^2 - g \Psi' \theta e_{w_y} e_{w_x} + g \Psi' \theta e_{w_y} e'_{w_x} + (g(g+1) \Psi^2 \theta - g(g+1) \Psi^2 \theta^2 - \\ &\left. g^2 \Psi^2 \theta) e_{w_x} e'_{w_x} \right\} + k_2 \mu_x \{ e'_{w_x} - e_{w_x} + g \Psi' \theta e_{w_x}^2 + g \Psi' \theta e'_{w_x} - 2g \Psi' \theta e_{w_x} e'_{w_x} \}. \end{aligned} \quad (3)$$

Taking expectation on both sides, we get the Bias as

$$\text{Bias}(t_{IR}) = -\mu_y + k_1 D + k_2 E. \quad (4)$$

Squaring both sides of Eq. (4) and taking expectation we get

$$\text{MSE}(t_{IR}) = \mu_y^2 + k_1^2 A + k_2^2 B + 2k_1 k_2 C - 2k_1 D - 2k_2 E, \quad (5)$$

where

$$\begin{aligned} A &= \mu_y^2 \{ 1 + f C_{w_y}^2 + (g^2 \Psi^2 \theta^2 + g(g+1) \Psi^2 \theta^2) f C_{w_y}^2 - g \Psi^2 \theta^2 (2g+1) f' C_{w_x}^2 - \\ &4g \Psi \theta f \rho_{w_x w_y} C_{w_x} C_{w_y} + 4g \Psi \theta f' \rho_{w_x w_y} C_{w_x} C_{w_y} \}, B = \mu_y^2 \{ f C_{w_x}^2 - f' C_{w_x}^2 \}, \end{aligned}$$

$$C = \mu_y \mu_x \left\{ 2g\psi\theta f C_{wx}^2 - 2g\psi\theta f' C_{wx}^2 - (f - f')\rho_{w_x w_y} C_{w_x} C_{w_y} \right\},$$

$$D = \mu_y^2 \left\{ 1 + \frac{g(g+1)}{2} \psi^2 \theta^2 f C_{wx}^2 - \frac{g(g+1)\psi^2 \theta^2}{2} f' C_{wx}^2 - (f - f')\rho_{w_x w_y} C_{w_x} C_{w_y} \right\},$$

$$E = \mu_y \mu_x (f - f') g \psi \theta f C_{wx}^2 \text{ and } \epsilon = \frac{1}{\mu_y}.$$

Partially differentiating Eq. (5) with respect to k_1 and k_2 and equating the resultant to zero, we obtain the optimum values as

$$k_{1 \min} = \frac{BD - CE}{AB - C^2}. \quad (6)$$

$$k_{2 \min} = \frac{AE - CD}{AB - C^2}. \quad (7)$$

Using Eqs. (6) and (7) in Eq. (5), we obtain the minimum MSE of the proposed class t_{IR} as

$$MSE(t_{IR}) = \mu_y^2 + k_{1 \min}^2 A + k_{2 \min}^2 B + 2k_{1 \min} k_{2 \min} C - 2k_{1 \min} D - 2k_{2 \min} E, \quad (8)$$

3 | Empirical Study Using Artificial Data

In this section, we conduct several empirical studies to demonstrate the efficiency of the new estimators developed from the proposed improved ratio type generalized class over similar competing existing estimators. For the empirical studies, the R programming language has been used. Using the model $Y_i = \frac{3}{4}x_i + e_i$ where $e \sim N(0, x_i)$ and the auxiliary variable X has been taken from Thompson [22] three different populations for the empirical study are generated. Using the condition $C = \{y \neq 0\}$ the networks are obtained and the transformed population is created. The y -values generated have been provided in the Appendix.

Since this paper focusses on Strategy I only i.e., when the second phase sample of size n is drawn from the first phase sample of size n' where $n \leq n'$, we proceed further to calculate the MSEs. The results are presented in the form of PREs in Table 1.

Table 1. Description of populations.

Population I	Population II	Population III
$C^2 = 25.97873$	$C^2 = 37.17892$	$C^2 = 97.59925$
w_y	w_y	w_y
$C^2 = 10.71544$	$C^2 = 10.71544$	$C^2 = 10.71544$
w_x	w_x	w_x
Coefficient of skewness $\beta_1(w_x) = 3.045973$	Coefficient of skewness $\beta_1(w_x) = 3.045973$	Coefficient of skewness $\beta_1(w_x) = 3.045973$
Coefficient of kurtosis $\beta_2(w_x) = 7.649341$	Coefficient of kurtosis $\beta_2(w_x) = 7.649341$	Coefficient of kurtosis $\beta_2(w_x) = 7.649341$
$S^2 = 1.1857$	$S^2 = 1.286855$	$S^2 = 0.666848$
w_y	w_y	w_y
$S^2 = 7.117463$	$S^2 = 7.117463$	$S^2 = 7.117463$
w_x	w_x	w_x
$\rho_{w_y w_x} = 0.7468$	$\rho_{w_y w_x} = 0.5087345$	$\rho_{w_y w_x} = 0.4706$
$N = 400$	$N = 400$	$N = 400$
$n' = 145$	$n' = 140$	$n' = 140$
$n = 140$	$n = 130$	$n = 130$

Table 2. Obtained PREs in respective populations.

Estimators	PRE in Population I	PRE in Population II	PRE in Population III
t_{Th}	100	100	100
t_r	102.9877	102.807	102.1856
t_p	93.21635	91.88633	95.72816
t_{g_1}	103.0493	102.8158	102.4005
t_{g_2}	103.0493	102.8158	102.4005
t_{g_3}	103.0493	102.8158	102.4005
t_{g_4}	103.0493	102.8158	102.4005
t_{g_5}	103.0493	102.8158	102.4005
t_{g_6}	103.0493	102.8158	102.4005
t_{G_1}	115.111	122.1203	153.0786
t_{G_2}	115.1108	122.1203	153.0771
t_{G_3}	115.1109	122.1204	153.0774

4 | Estimating Average Number of Thorny Plants

In this section, we use the developed and existing estimators to estimate the average number of thorny plants using the data set of Latpate and Krishnasagar [18]. As pointed out by Latpate and Krishnasagar [18], aluminum content of the soil and existence of thorny plants are negatively correlated therefore the ratio estimator by Raghav et al. [15] is expected to not perform well. We use all of the discussed estimators existing and new developed to estimate the average number of thorny plants using the percentage of Aluminum as auxiliary informations. To estimate the average number of thorny plants with the two stage ACS design, we use $C = \{y_i > 0\}$ and obtain a first phase sample of size $n' = 150$ from a population size $N = 400$ and estimate μ_X . Since we are using Strategy I, we draw the second phase sample of size $n = 140$ and use all the estimators, developed and existing to estimate the average number of thorny plants. The description of the dataset and the performance of the estimators are in Table 3.

Table 3. Obtained PREs in respective populations.

Description of Population	Estimators	MSEs	PREs
$C^2 = 6.36$	t_{Th}	5.5969	100
w_y			
$C^2 = 0.16$	t_r	5.7385	97.53
w_x			
$\beta_1(w_x) = -0.31$	t_p	5.4842	102.05
$\beta_2(w_x) = -0.94$	t_{g1}	5.3177	105.25
$S^2 = 1204.48$	t_{g2}	5.3177	105.25
w_y			
$S^2 = 210.16$	t_{g3}	5.3177	105.25
w_x			
$\rho_{wywx} = -0.70$	t_{g4}	5.3177	105.25
$N = 400$	t_{g5}	5.3177	105.25
$n' = 150$	t_{g6}	5.3177	105.25
$n = 140$	t_{G1}	5.1726	108.20
$MR = 34.02$	t_{G2}	5.1726	108.20
$HL = 36.64$	t_{G3}	5.1726	108.20

5 | Conclusion

In this paper our focus was to extend the Gupta and Shabbir [16] class by making it more generalized and study it in the two phase ACS design under the transformed population approach. Motivated by Olkin [19], Qureshi and Hanif [20], Singh et al. [17], [21], we proposed a new class tG by making the Gupta and Shabbir

[16] class more generalized and studied it in the two phase ACS design. From the proposed class tG, we developed some new estimators which are presented in Section 2.

Using empirical studies on three different populations, we compared the performance of the new developed estimators with several existing competing estimators of Raghav et al. [15]. The results are tabulated in *Table 1*. From these results we observed that the performance of the developed estimators tG1 – 3 is far better than the existing competing estimators. Further, we used the new developed estimators and the existing competing estimators to estimate the average number of thorny plants and the results of this study are tabulated in *Table 2*. In this study, similar results can be seen and therefore we conclude that the new developed estimators are better than the existing competing ones and when this sampling design is to be used, the new developed estimators may be used subject to availability of the parameters of auxiliary variable which for the new developed estimators have been taken to be known. For future research, the class can be studied using the concept of intrinsic function to simultaneously optimize k_1 , k_2 , θ and g .

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Conflicts of Interest

The authors have no conflict of interest.

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Appendix A

Table A1. Existing competing estimators.

Description of Population	Estimators
$t_{Th} = \frac{1}{n} \sum_{i=1}^n w_{y_i}$ [1]	$f S_{wy}^2$
$t_r = \bar{w}_y \frac{\mu_x^*}{\bar{w}_x}$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 (C_{wy}^2 + C_{wx}^2 -$
$t_p = \bar{w}_y \frac{\bar{w}_x}{\mu_x^*}$ [15]	$2\rho_{w_x w_y} C_{wy} C_{wx}) \left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 (C_{wy}^2 +$
$t_{g1} = \bar{w}_y \left(\frac{MR \mu_x^* + HL}{\eta_{lopt} (MR \bar{w}_x + HL) + (1 - \eta_{lopt}) (MR \mu_x^* + HL)} \right)^1$ [15]	$C_{wx}^2 + 2\rho_{w_x w_y} C_{wy} C_{wx})$
$t_{r2} = \bar{w}_y \left(\frac{MR \mu_x^* + \beta_1(w_x)}{\eta_{lopt} (MR \bar{w}_x + \beta_1(w_x)) + (1 - \eta_{lopt}) (MR \mu_x^* + \beta_1(w_x))} \right)^1$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 C_{wy}^2 (1 - \rho_{w_y w_x}^2)$
$t_{r3} = \bar{w}_y \left(\frac{MR \mu_x^* + \beta_2(w_x)}{\eta_{lopt} (MR \bar{w}_x + \beta_2(w_x)) + (1 - \eta_{lopt}) (MR \mu_x^* + \beta_2(w_x))} \right)^1$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 C_{wy}^2 (1 - \rho_{w_y w_x}^2)$
$t_{r4} = \bar{w}_y \left(\frac{HL \mu_x^* + \beta_1(w_x)}{\eta_{lopt} (HL \bar{w}_x + \beta_1(w_x)) + (1 - \eta_{lopt}) (HL \mu_x^* + \beta_1(w_x))} \right)^1$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 C_{wy}^2 (1 - \rho_{w_y w_x}^2)$
$t_{r5} = \bar{w}_y \left(\frac{HL \mu_x^* + \beta_2(w_x)}{\eta_{lopt} (HL \bar{w}_x + \beta_2(w_x)) + (1 - \eta_{lopt}) (HL \mu_x^* + \beta_2(w_x))} \right)^1$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 C_{wy}^2 (1 - \rho_{w_y w_x}^2)$
$t_{r6} = \bar{w}_y \left(\frac{HL \mu_x^* + MR}{\eta_{lopt} (HL \bar{w}_x + MR) + (1 - \eta_{lopt}) (HL \mu_x^* + MR)} \right)^1$ [15]	$\left(\frac{1}{n'} - \frac{1}{N}\right) \mu_y^2 C_{wy}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right) \mu_y^2 C_{wy}^2 (1 - \rho_{w_y w_x}^2)$

0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	5.035436	0.1076227	1.260711	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.4769653	63.1953717	4.378328	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	-0.1187378	-2.1372071	-2.6869822	-1.623478	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	-0.2410239	-0.6271539	-1.583597	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.730419	-4.8540662	22.7441947	2.890208	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
2.552982	-17.2491230	-17.4884486	6.403189	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	2.0020025	3.2167183	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000	0.000000	0.000000	0	0	0
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0	0	0	0	0	0	0.000000	0.000000					

Fig. A1. Generated γ -values of population 1.

0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	23.5741456	-7.732837	0.9264182	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	-11.6830157	-2.899202	5.0544067	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	-1.180232	-0.3010663	3.371058	-2.7570083	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	1.136070	0.8221523	-1.3649024	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
1.103161	-15.280092	0.3210738	5.2106956	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
-1.357273	-30.447913	29.7788156	0.2878866	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	1.651176	-2.4049225	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	-1.148564	0.3764652	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	7.0621268	34.743631	9.6954949	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	-0.9290897	20.5645316	-13.223210	13.7184883	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	2.8270994	3.917543	0.9845512	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0
0.000000	0.000000	0.0000000	0.0000000	0.000000	0.0000000	0	0	0	0	0	0	0.000000	0.0000000	0.000000	0.0000000	0	0	0

Fig. A2. Generated y-values of population 2.

0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	10.68495	0.2224865	0.5843418	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	28.68901	22.1197015	2.9584996	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.2505814	-0.87141	-12.9276881	-1.6722522	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.1547305	-1.2302333	1.522983	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
-1.6096717	-5.0102061	7.2777808	-5.208132	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.8540489	-0.9683967	-13.3597798	4.167680	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	-4.1911109	2.2129749	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	-1.7447336	0.5931299	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	2.060744	23.6971021	0.4658569	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.8070798	3.698443	-41.9131488	3.3480259	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	8.873167	-0.6594561	-0.8134932	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0
0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0	0	0	0	0	0	0.0000000	0.00000	0.0000000	0.0000000	0	0	0

Fig. A3. Generated γ -values of population 3.